Violation of Porod law in a freely cooling granular gas in one dimension

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We study a model of freely cooling inelastic granular gas in one dimension, with a restitution coefficient which approaches the elastic limit below a relative velocity scale δ. While at early times ($t \ll \delta^{-1}$) the gas behaves as a completely inelastic sticky gas conforming to predictions of earlier studies, at late times ($t \gg \delta^{-1}$) it exhibits a new fluctuation dominated phase ordering state. We find distinct scaling behavior for the (i) density distribution function, (ii) occupied and empty gap distribution functions, (iii) the density structure function and (iv) the velocity structure function, as compared to the completely inelastic sticky gas. The spatial structure functions (iii) and (iv) violate the Porod law. Within a mean-field approximation, the exponents describing the structure functions are related to those describing the spatial gap distribution functions.

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Flowing granular media exhibit varied physical phenomena [1, 2]. A simple, well studied model that captures many features of flowing granular systems is a gas of particles undergoing inelastic collisions [2, 3, 4, 5, 6, 7, 8, 9, 10]. The system may be externally driven or allowed to cool freely. However, real granular gases have *elastic* collisions when relative velocity of particles approach zero [11]. Thus, a realistic model of cooling granular gas should have a relative velocity dependent restitution coefficient [11]. In this paper we focus on such a model.

In general, a system freely relaxing to an ordered state has a macroscopic length scale $\mathcal{L}(t)$ increasing with time t [12]. In addition, for usual phase ordering systems, the presence of a dominant $\mathcal{L}(t)$ results in a robust scaling law called the Porod law [12, 13]. For scalar order parameters, the Porod law states that the scaled structure function $S/\mathcal{L}^d \sim (k\mathcal{L})^{-\theta}$ for large $k\mathcal{L}$, with $\theta = 2$ in one dimension. Contrary to this clean scenario, in certain driven systems [14, 15, 16, 17], an unusual phase ordering was observed. These systems have a macroscopic coarsening length scale $\mathcal{L}$ but the domain lengths have a power-law distribution with a large negative power. This leads to a violation of the Porod law with $\theta \neq 2$. We shall refer to such systems as fluctuation-dominated phase ordering (FDPO) systems. We demonstrate below that a coarsening granular gas too has such an unusual FDPO state.

The issue of an inelastic gas showing coarsening and phase ordering has been addressed in many earlier publications on the subject [5, 6, 7, 8, 10, 18, 19]. It is now well-known that both the freely cooling inelastic gas ($0 < r < 1$) and the sticky gas ($r = 0$) undergo coarsening with a growing length scale $\mathcal{L}(t) \sim t^{1/z}$, with $z = 3/2$ in one dimension [3, 5, 6]. The sticky gas problem in one dimension can be solved exactly and is known to be equivalent to the inviscid Burgers equation [5, 20]. From this equivalence, the structure functions of the sticky gas can be inferred to obey the Porod law [20].

Numerical studies have tried to relate the behavior of the inelastic gas, to the sticky gas. It was shown that at large times the decay of total energy of the inelastic gas is identical to the sticky gas [6]. Moreover, other quantities like $\mathcal{L}(t)$ and velocity distribution function have the same scaling form as that of the sticky gas. This suggested that for any deviation from the elastic limit, the large time scaling behavior crosses over to that of the sticky gas and that the underlying continuum equation is the inviscid Burgers equation [6].

In this paper, we show that if a granular gas is modeled as a gas of particles having a velocity dependent restitution coefficient, interesting new physics appear. In the late time regime, the equivalence with the sticky gas breaks down and the system exhibits a FDPO state. For our granular gas model, velocity dependence of the restitution coefficient is chosen to be:

$$r = (1 - r_0)\exp\left(-|v_{\rm rel}/\delta|^{\sigma}\right) + r_0. \qquad (1)$$

For relative velocity $v_{\rm rel} \ll \delta$, $r \to 1$, and for $v_{\rm rel} \gg \delta$, $r \to r_0 < 1$, mimicking the experimental scenario [11]. The parameter σ determines how sharply the crossover from r_0 to 1 happens across the crossover scale δ. While experiments [11] suggest a wide range of values for σ, kinetic theory studies estimate $\sigma = 1/5$ [21]. We note that taking first the limit $\sigma \to \infty$ and then $\delta \to 0$, this model becomes the same as the model studied in Ref. [6].

Our main result is that the cooling granular gas has a time scale $t_1 \sim \delta^{-1}$, such that the density distribution function, and various *spatial* distribution functions show a complete change of behavior across it. Yet at the same time, the total energy $E(t)$ decay as $\sim t^{-2/3}$ throughout, without any signature of change across the timescale t_1. For $t \ll t_1$, we find that the granular gas behaves as a sticky gas (as in [6]). But, for late times $t_1 \ll t \ll t_2$ (where $t_2 \sim \delta^{-3}$), the phase ordering is distinct from the sticky gas. In particular, the density-density and velocity-velocity structure functions show violation of the

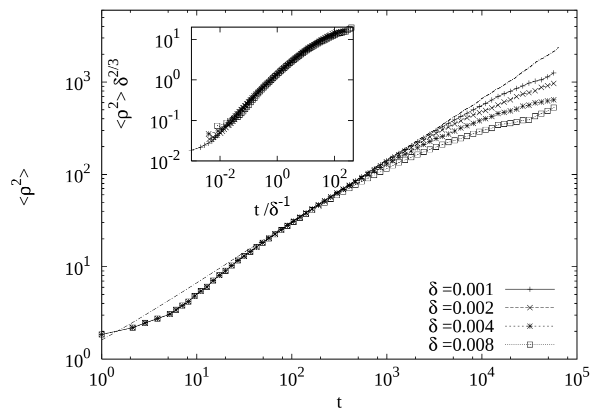

FIG. 1: The variation of $\langle \rho^2 \rangle$ with time t is shown for different values of δ when $\sigma = 3.0$. The dashed line corresponds to the sticky gas. Inset: The curves collapse when t is scaled by δ^{-1}.

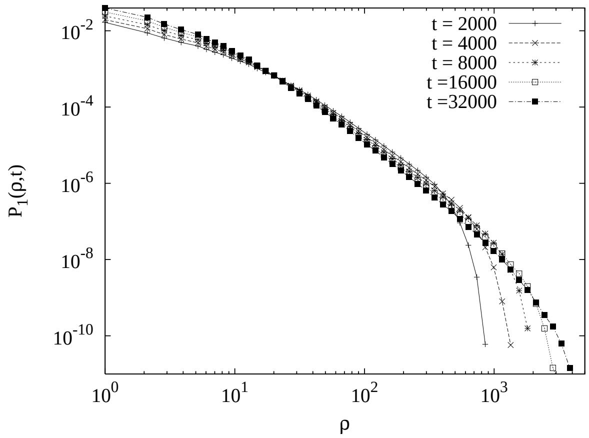

FIG. 2: The variation of $P_1(\rho, t)$ with ρ is shown for different times.

Porod law. We note that the scale t_2, beyond which all collisions are elastic and $E(t)$ stops decreasing, is easily understood in terms of the velocity scaling law [6] – $v(t_2) \sim t_2^{-1/3} \sim \delta$ implying $t_2 \sim \delta^{-3}$. On the contrary, the interesting scale t_1 that we find is much smaller.

We now define our model more precisely. We consider N point particles of equal mass on a ring of length L. Initially, the particles are distributed randomly in space with their velocities drawn from a normal distribution. The particles undergo inelastic, momentum conserving collisions such that when two particles with velocity u_i and u_j collide, the final velocities u'_i and u'_j are given by:

$$u'_{i,j} = u_{i,j} \left(\frac{1-r}{2} \right) + u_{j,i} \left(\frac{1+r}{2} \right). \tag{2}$$

We define coarse-grained densities and velocities for the granular gas as follows [9]. At any point of time the system is divided into N equally sized spatial boxes. The total number of particles in the ith ($i = 1, 2, \ldots, N$) box defines the mass density ρ_i. The velocity v_i is defined as the sum of the velocities of the particles in box i. For the sticky gas it suffices to talk about distributions of masses and velocities of individual particles.

We have done an event driven molecular dynamics simulation for system sizes L, ranging from $20000 - 50000$ (in units of inter particle spacing). The particle density is set to 1 throughout. The system was evolved up to time $t = 32000 - 64000$ (in units of initial mean collision time). For these times $\mathcal{L}(t) \ll L$. Simulations were done for different σ (namely 3, 4, 5, 10 and ∞), δ (namely $0.001 - 0.01$), and r_0 (namely 0.1, 0.5 and 0.8) values. There was no qualitative difference found for the various sets of these parameter values. Hence, we choose a specific set for the data presented below – $r_0 = 0.5$, $\sigma = \infty$, $\delta = 0.01$ (unless otherwise mentioned). Whenever there is a quantitative dependence on σ, we mention it.

The existence of the time scale t_1 can be seen by examining $\langle \rho^2 \rangle$, where ρ is the density. In Fig. 1 we show the time dependence of $\langle \rho^2 \rangle$ for systems with same $\sigma = 3$ but different δ values. At early times $\langle \rho^2 \rangle \sim t^{2/3}$ as for the sticky gas. The departure from the sticky gas curve happens at a timescale t_1 which increases with decreasing δ. The curves collapse when t is scaled by δ^{-1} (see inset). We have checked that this dependence is independent of σ, i.e., $t_1 \sim \delta^{-1}$ for all σ.

Currently we do not have a deeper understanding of the timescale t_1. Intriguingly, we find no signature of t_1 in the decay of the total energy $E(t) \sim t^{-2/3}$, which is related to the second moment of the velocity distribution. In this paper, we concentrate on the density distribution and various *spatial* distribution functions. The spatial distributions include the empty and occupied gap distribution functions, and the density and velocity structure functions. For the early time regime $t \ll t_1$, these quantities are numerically equivalent to the corresponding quantities of the sticky gas [22]. In this paper, we focus on the late time regime $t \gg t_1$, where there is deviation from the sticky gas.

Let $P_1(\rho, t)$ be the probability that a box has mass density ρ at time t. In Fig. 2, the variation of $P_1(\rho, t)$ with ρ is shown for different times. For small ρ ($\rho < 10$), $P_1(\rho, t)$ increases with time to a nonzero constant. For intermediate ρ the slope increases to a constant. The cutoff increases to infinity with time. From these features, we conclude that when $t \gg 1$, $P_1(\rho, t)$ approaches a non-zero time independent power law distribution, i.e.

$$\lim_{t \gg t_1} \lim_{L \to \infty} P_1(\rho, t) \sim \rho^{-\gamma_1}. \tag{3}$$

Conservation of density ($\langle \rho \rangle = 1$) implies that $\gamma_1 > 2$. Consistent with this, we find that the power law for the curves at various times extrapolate asymptotically to $\gamma_1 \simeq 2.83$. The cutoff $\rho_{max}(t)$ scales as $\mathcal{L}(t) \sim t^{\beta}$ with $\beta = 2/3$. There are strong corrections to scaling,

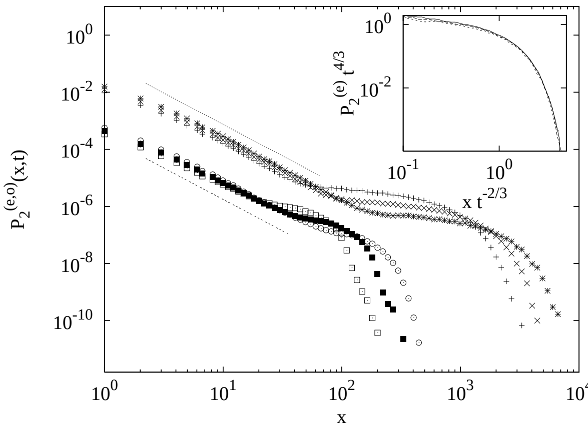

FIG. 3: The variation of $P_2^{(e,o)}(x,t)$ with x is shown for times $t = 16000, 32000, 64000$ (the larger times have larger cutoffs). $P_2^{(o)}(x,t)$ is shifted downwards for clarity. The straight lines have power -2.2. Inset: The plateaus of $P_2^{(e)}(x,t)$ have the scaling form $t^{-4/3}f(xt^{-2/3})$.

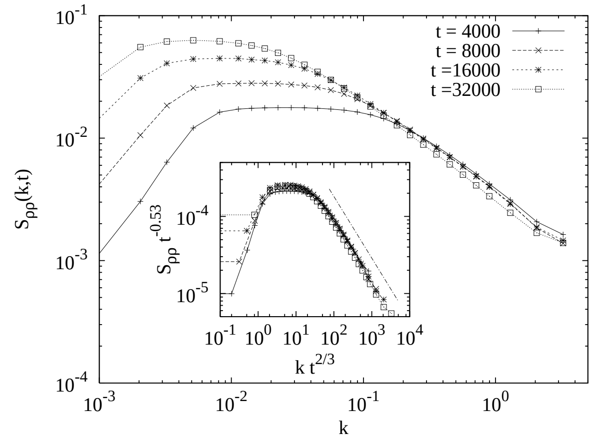

FIG. 4: The variation of $S_{\rho\rho}(k,t)$ with k is shown for different times. The inset shows the data scaled as in Eq. (8). The straight line has power -0.80.

as is clear from the change in apparent slope for different times. Hence, it is not possible to obtain data collapse by scaling unless one measures $P_1(\rho,t)$ for even larger densities and times. The value of γ_1 depends on the value of σ. For σ varying from 3 to ∞, γ_1 varies from 2.30 to 2.83.

The data of Fig. 2 for our granular gas implies a scenario in which small density clusters do not get depleted from the system. This is in contrast to the sticky gas, where the probability of the mass clusters , $P_{1s}(m,t) \sim m^{-1/2}t^{-1/3}$ for $m \ll t^{2/3}$ [5] decays to zero with time. The coarsening process in the sticky gas is one of pure aggregation, transferring mass from smaller to larger mass scales. On the other hand, our model at late times effectively shows a combined aggregation and fragmentation dynamics. This simply comes as the "elastic" collisional break-ups at late times (and small velocities) compete with the aggregation due to inelasticity. The effective rates are such that mass loss and gain at small scales are balanced out.

Further support to the above picture comes from the inter particle and inter hole gap distribution functions. Let $P_2^{(e,o)}(x,t)$ be the probability of finding a gap of exactly x empty(e) or occupied(o) boxes. The variation of $P_2^{(e,o)}(x,t)$ with x for different times is shown in Fig. 3. For small masses, they decay as a power law with power $\gamma_2 \simeq -2.2$. For large masses, $P_2^{(o)}(x,t)$ has a plateau eventually cutoff at scales $x \sim \mathcal{L}(t)$. The shape of the plateau and cutoff is reminiscent of the the number distribution of gaps in the sticky gas which have the form $N_{2s}(x,t) = t^{-4/3}f_{2s}(x/\mathcal{L}(t))$ with $f_{2s}(z) \to 1$ for $z \ll 1$ [5]. The large x of of $P_2^{(o)}(x,t)$ scales exactly as $N_{2s}(x,t)$ (see inset of Fig. 3). Since $\int dx N_{2s} \sim t^{-2/3}$, the area under the plateau will eventually go to zero. So while the big gaps in our model scale as that of the sticky gas, the more important fact is that it has predominantly small gaps associated with the following power law:

$$\lim_{t \gg t_1} \lim_{L\to\infty} P_2^{(e,o)}(x,t) \sim x^{-\gamma_2}. \tag{4}$$

We have checked that the exponent γ_2 has no discernible dependence on σ.

The inter particle gap distribution has a bearing on the density-density and velocity-velocity correlation functions. In particular, the large power γ_2 suggests that the ordering process will be affected by the abundance of smaller gaps and Porod law could be violated. For the sticky gas, numerical results [20, 22] confirm that Porod law holds for both density-density and velocity-velocity correlation functions. We now discuss the case of our granular gas and show that it is different.

Let $C_{\rho\rho}(x,t) = \langle \rho_i(t)\rho_{i+x}(t)\rangle$ be the equal time density-density correlation function. The structure function $S_{\rho\rho}(k,t)$ is the Fourier transform of $C_{\rho\rho}(x,t)$. Similarly we define the equal time velocity-velocity correlation function as $C_{vv}(x,t) = \langle v_i(t)v_{i+x}(t)\rangle$, with its corresponding structure function $S_{vv}(k,t)$. We show that the structure functions can be expressed in terms of the gap distribution through a mean field approximation. $C_{\rho\rho}(x)$ is approximately equal to density square times the probability there is a non-zero density at x given there is a non-zero density at 0. Thus (setting $\rho = 1$), $C_{\rho\rho}(x) \approx \sum_{n=0}^{\infty} p_{2n}(x)$, where $p_{2n}(x)$ is the probability of having exactly n empty gaps between 0 and x with 0 and x being occupied. Let $\tilde{C}_{\rho\rho}(s)$ and $\tilde{P}_2^{(o,e)}(s)$ be the Laplace Transforms of $C_{\rho\rho}(x)$and $P_2^{(o,e)}(x)$ respectively. Approximating joint distributions by products of individ-

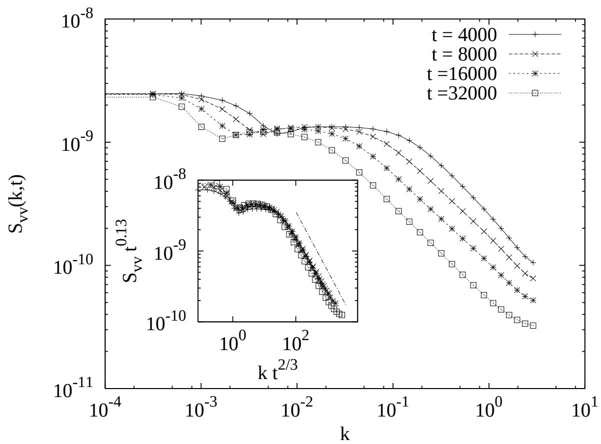

FIG. 5: The variation of $S_{vv}(k,t)$ with k is shown for different times. The inset shows the data scaled as in Eq. (9). The straight line has power -0.80.

ual distributions [23], we obtain

$$\tilde{p}_{2n}(s) = \frac{\left[1-\tilde{P}_2^{(o)}\right]^2 \tilde{P}_2^{(e)} \left[\tilde{P}_2^{(o)}\tilde{P}_2^{(e)}\right]^{n-1}}{\langle x\rangle_o s^2}, \; n \geq 1, \quad (5)$$

$$\tilde{p}_0(s) = \frac{1}{s} - \frac{1-\tilde{P}_2^{(o)}}{s^2\langle x\rangle_o}, \quad (6)$$

where $\langle x\rangle_o = \int x P_2^{(o)}(x)dx$. Eqs. (5) and (6) give

$$\tilde{C}_{\rho\rho}(s) = \frac{1}{s} - \frac{[1-\tilde{P}_2^{(o)}(s)][1-\tilde{P}_2^{(e)}(s)]}{\langle x\rangle_o s^2[1-\tilde{P}_2^{(e)}(s)\tilde{P}_2^{(o)}(s)]}. \quad (7)$$

The correlation function $C_{vv}(x)$ is exactly the same except for a factor of $\langle v^2\rangle$. Thus, $\tilde{C}_{vv}(s) \sim v_t^2 \tilde{C}_{\rho\rho}(s)$.

Now, $1-\tilde{P}_2^{(e,o)}(s) \sim s^{\gamma_2-1}$. Also, $\langle x\rangle_o \sim \mathcal{L}^0$ since $\gamma_2 > 2$. This implies that $\tilde{C}_{\rho\rho}(s) \sim s^{\gamma_2-3}$, where we have ignored the s^{-1} which contributes to $\delta(k)$ in the Fourier transform. Thus, $S_{\rho\rho}(k)$ will have the scaling form

$$S_{\rho\rho}(k) = \mathcal{L}^{3-\gamma_2} f_3(k\mathcal{L}), \quad (8)$$

where $f_3(x) \sim x^{-\theta_\rho}$ for $x \gg 1$, with $\theta_\rho = 3-\gamma_2$. Using $\langle v^2\rangle \sim \mathcal{L}^{-1}$ [6], we obtain

$$S_{vv}(k) = \mathcal{L}^{2-\gamma_2} f_4(k\mathcal{L}), \quad (9)$$

where $f_4(x) \sim x^{-\theta_v}$ for $x \gg 1$ with $\theta_v = 3-\gamma_2$. Since $\gamma_2 \approx 2.20$, $\theta_v = \theta_\rho \approx 0.80$ which is very different from 2.0 seen in Porod law in one dimension.

The mean field approximation gives a good description of the actual problem. In Fig. 4 we show the variation of $S_{\rho\rho}(k,t)$ with k. The data collapses onto one curve when scaled as in Eq. (8) [see inset of Fig. 4]. The scaling function $f_3(z)$ varies as $z^{-0.8}$ for large z. In Fig. 5, the variation of $S_{vv}(k,t)$ with k is shown. Again a good collapse is obtained when the data is scaled as in Eq. (9) such that $f_4(z) \sim z^{-0.8}$ for large z. We found that the exponents θ_ρ and θ_v have no dependence on σ.

To summarize, we studied a model of freely cooling granular gas with velocity dependent restitution coefficient. We showed the existence of a time scale $t_1 \sim \delta^{-1}$ beyond which the system deviates from the sticky gas behavior. The effective dynamics in this regime is one of aggregation and fragmentation. As a result, the spatial distribution functions change their forms drastically. We found new power law exponents associated with the density distribution function and empty and occupied gap distribution functions. The two-point spatial correlation functions violate Porod law and the structure function decay exponent is $\simeq 0.8$ instead of the usual value 2. This deviation and the existence of power laws in the one-point functions indicate that the phase ordering is dominated by large scale fluctuations. We hope that experiments on cooling granular gases in quasi one-dimension would find the deviation in Porod law predicted by us.

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